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The Hopf algebra of renormalization, normal coordinates and Kontsevich deformation quantization

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Abstract

Using normal coordinates in a Poincaré–Birkhoff–Witt basis for the Hopf algebra of renormalization in perturbative quantum field theory, we investigate the relation between the twisted antipode axiom in that formalism, the Birkhoff algebraic decomposition and the universal formula of Kontsevich for quantum deformation.

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1. Introduction

A regular feature in frontier physics and mathematics has been the passage from commutative to non-commutative structures [1], and deformation quantization has been a major factor in this trend (for a nice review of the genesis, developments and major metamorphoses in this field we refer the reader to the paper by Dito and Sternheimer in [2]). An important contribution to the metamorphosis of deformation quantization has been the work of Kontsevich [3] and the proof therein of his Formality theorem, which allowed us to establish the existence of star associative products on general Poisson differentiable manifolds.

The most obvious example of the relevance in physics of deformation quantization is the Moyal product, based on a constant Poisson structure, and which exhibits the passage from classical to quantum mechanics as a deformation of the pointwise product of smooth functions on $\mathbb{R}^d$ in the direction of the Poisson product. It is well known that the Moyal deformation operator is the exponential of a bi-differential.

Next in the order of complexity are the linear Poisson structures for which the paradigm is the Lie–Poisson bracket first introduced by S Lie himself, and latter rediscovered by F Berezin and A Kirillov. In fact, the analysis of a linear Poisson structure on $\mathbb{R}^d$ is equivalent to considering the vector space dual to a Lie algebra with the Poisson structure induced by the Lie bracket of the algebra. For this linear Poisson structure there are at least two canonical quantization deformations known: the universal formula of Kontsevich (equivalent to the Duflo star product [4]) and the one arising from the classical Baker–Campbell–Hausdorff (BCH) formula. The BCH product corresponds to the Gutt product [5] obtained from the product of elements in the universal enveloping algebra via the symmetrization operator, while in the Kontsevich construction (which for clarity purposes we review in section 5), each term in the product corresponds to a graph, associated with a poly-differential operator, and all graphs have a weight defined by the integration of a $2n$-form, where n is the set of edges of the graph.

The relation between these two quantizations has been considered recently by various authors [6–10]. It was shown by Kathotia [7] that the BCH formula is exactly that part of the Kontsevich formula consisting of all the admissible L-graphs without wheels and that the two quantizations are totally equivalent for the case of nilpotent Lie algebras.

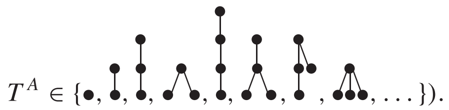
Based on the relation of the above-mentioned quantization deformations, the Hopf algebraic formulation of renormalization in perturbative quantum field theory (pQFT), first discovered by Kreimer [11] and further developed by Connes and Kreimer [12, 13], together with our introduction of the concept of normal coordinates in the Hopf algebra of renormalization [14], we shall show here that the Forest formula for renormalization in pQFT and Birkhoff's algebraic decomposition in that context can be interpreted for any renormalizable field theory, as a Kontsevich star-product deformation in the direction of the Lie–Poisson product, and where the Kontsevich poly-differential operator is an exponential of a sum of admissible prime L-graphs.

Indeed, the Connes–Kreimer formalism involves two Hopf algebras: the (commutative, but not co-commutative) Hopf algebra $\mathcal{H}_R$ generated by representatives of decorated rooted trees, and the (non-commutative, co-commutative) Hopf algebra Char$\mathcal{H}_R$ of the group of characters in duality with $\mathcal{H}_R$ and isomorphic to the universal enveloping algebra $U(\mathfrak{L})$ of a Lie algebra $\mathfrak{L}$. But, on one hand, the deformation quantization of the universal envelope of a Lie algebra corresponds to group multiplication via the BCH formula and, on the other hand, the use of normal coordinates in the construction of a basis for $\mathcal{H}_R$ introduces a group product and BCH formula in the definition of the coproduct for the normal coordinates. Consequently, since such a coproduct appears in the twisted antipode axiom for renormalization within the Connes–Kreimer and normal coordinates formalism (cf equation (29)), it becomes reasonable to expect a relation between renormalization in pQFT and the poly-differential symplectic operator of Kontsevich for quantum deformations in the case of a linear Poisson structure. The key point in this observation is that although the Lie algebra $\mathfrak{L}$ is not nilpotent all the wheels are null.

2. The Hopf algebra of renormalization, characters, infinitesimal characters and normal coordinates

As a basis for our discussion we shall make use of the normal coordinates for the Hopf algebra of renormalization, which we previously introduced in [14]. So in order to fix notation and make our presentation as self-contained as possible, we begin by reviewing some of the relevant results in that paper which we shall be making use of here.

One-particle irreducible superficially divergent Feynman diagrams in pQFT can be represented by decorated rooted trees (or sums of them for the case of overlapping divergences) which are finite, connected graphs without loops where every vertex has one incoming edge except for the root that has only outgoing edges. The decorations of the vertices are primitive diagrams (divergent but without subdivergences) [12]. Let $\mathcal{H}_R\ (m, \mathbf{1} = e1, \Delta, \epsilon, S)$ denote the graded commutative (but not co-commutative) Hopf algebra, over a field $\mathbb{K}$ of characteristic zero, generated by the rooted trees. By the Milnor-Moore theorem, there is a co-commutative Hopf algebra $\mathbf{G} = \mathrm{Char}\mathcal{H}_R$ in duality with $\mathcal{H}_R$, isomorphic to the universal enveloping algebra $U(\mathfrak{L})$ where $\mathfrak{L} = \partial\,\mathrm{Char}\mathcal{H}_R$ is a Lie algebra. $\mathbf{G}$ is the group of characters of $\mathcal{H}_R$ (algebra morphism under the convolution product $\langle \eta * \lambda, T^A\rangle := \langle \eta \otimes \lambda, \Delta T^A\rangle$, $\eta, \lambda \in \mathbf{G}$, with T^A a representative of an isomorphism class of rooted trees e.g.



(1)

Let Z_A denote the infinitesimal generators of $\mathfrak{L}$ indexed by rooted trees and defined by

$$\langle Z_A, T^B\rangle = \delta_A^B, \tag{2}$$

$$\langle Z_A, T^B T^C\rangle = \langle Z_A, T^B\rangle \epsilon(T^C) + \epsilon(T^B)\langle Z_A, T^C\rangle. \tag{3}$$

Since the coproduct in $U(\mathfrak{L})$ is dual to the product in $\mathcal{H}_R$ we have

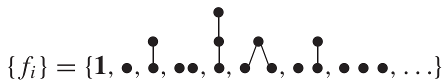
$$\begin{aligned} &\langle Z_A * Z_B, T^C\rangle = \langle Z_A \otimes Z_B, \Delta T^C\rangle = \sum_T n^T_{T^A T^B}\langle Z_T, T^C\rangle, \\ &Z_A, Z_B \in \partial Char\mathcal{H}_R, \end{aligned} \tag{4}$$

which defines a pre-Lie algebra on $\partial\,\mathrm{Char}\mathcal{H}_R$, and the Lie bracket

$$\begin{aligned} [Z_A, Z_B] &:= Z_A * Z_B - Z_B * Z_A = \sum_T \left(n^T_{T^A T^B} - n^T_{T^B T^A}\right) Z_T \\ &\equiv \sum_T f^T_{T^A T^B} Z_T, \end{aligned} \tag{5}$$

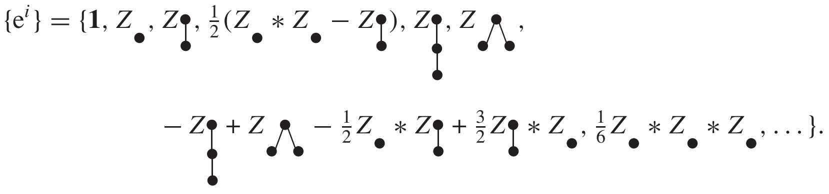
where $n^T_{T^A T^B}$ is the number of simple cuts on T that produce the sub-trees T^A and T^B, with T^B containing the root of T. The last equality in (5) defines the structure constants $f^T_{T^A T^B}$ of $\mathfrak{L}$.

Now, if e.g.



(6)

is a given Poincaré–Birkhoff–Witt basis for $\mathcal{H}_R$, we can obtain a dual basis $\{e^i\}$ for the enveloping algebra $U(\mathfrak{L})$ by adjoining to the above Z's polynomials in them (via the convolution product given by (4)) with $\langle e^i, f_j\rangle = \delta^i_j$. For the basis dual to (6) we get (for the case of vertices with the same decoration)



(7)

Clearly the calculation of the elements of the basis (7) becomes increasingly more complicated with increasing degree (number of vertices in the trees). We can change however the basis for $U(\mathfrak{L})$ to the simpler one

$$\{e^{i'}\} = \{\mathbf{1}, Z_A, Z_A * Z_B, \ldots\} = \{\mathbf{1}, Z_{\bullet}, \ldots, Z_{\bullet} * Z_{\bullet}, \ldots, \}. \tag{8}$$

In order to construct its dual, let $\tilde{\psi}^A$ be new coordinates centred at the origin and indexed by rooted trees. Choose then a new linear basis with the following ordering:

$$\{f_{i'}\} = \{\mathbf{1}, \tilde{\psi}^A, \tilde{\psi}^A \tilde{\psi}^B, \ldots\}. \tag{9}$$

Since $\{e^{i'}\}$ and $\{f_{i'}\}$ are by construction dual to each other, the canonical tensor

$$\mathbf{C} = \sum_i f_{i'} \otimes e^{i'} = e_*^{\tilde{\psi}^A \otimes Z_A}, \tag{10}$$

where the $*$-exponential, defined by

$$e_*^x = \sum_{i=0}^{\infty} \frac{1}{i!} \underbrace{x * \cdots * x}_{i \text{ factors}}, \qquad x \in \mathcal{U}_1, \tag{11}$$

acts as an identity on T^A, i.e.

$$\left\langle e_*^{\tilde{\psi}^B \otimes Z_B}, \mathrm{id} \otimes T^A \right\rangle = T^A. \tag{12}$$

Hence

$$T^A = \sum_{m=0}^{\infty} \frac{1}{m!} \tilde{\psi}^{B_1} \cdots \tilde{\psi}^{B_m} \langle Z_{B_1} * \cdots * Z_{B_m}, T^A \rangle. \tag{13}$$

Moreover, since

$$\left\langle Z_{B_1} * \cdots * Z_{B_m}, T^A \right\rangle = \left\langle Z_{B_1} \otimes \cdots \otimes Z_{B_m}, \Delta^{m-1}(T^A) \right\rangle, \tag{14}$$

where the higher powers of the convolution product are defined iteratively by

$$\Delta^{(0)} = \mathrm{id}, \qquad \Delta^{(m)} := (\mathrm{id} \otimes \Delta^{(m-1)}) \circ \Delta, \tag{15}$$

we have that

$$T^A = \tilde{\psi}^A + \sum_{m=2}^{\infty} \frac{1}{m!} \tilde{\psi}^{B_1} \cdots \tilde{\psi}^{B_m} n^A_{B_1 \cdots B_m}, \tag{16}$$

with

$$n^A_{B_1 \cdots B_m} = n^A_{B_1 R_1} n^{R_1}_{B_2 R_2} \cdots n^{R_{m-2}}_{B_{m-1} B_m}, \tag{17}$$

and summation over repeated indices understood throughout.

The relation between the above two sets of generators can be expressed concisely by

$$e_*^{\tilde{\psi}^A \otimes Z_A} = T^B \otimes Z_B. \tag{18}$$

It is not difficult to see from (13) that the matrix relating the Poincaré–Birkhoff–Witt bases $\{f_i\}$ and $\{f_{i'}\}$ is upper triangular with units along the diagonal, and therefore invertible (cf [14]). Consequently the normal coordinates $\tilde{\psi}^A$ are polynomials in terms of rooted trees with the linear part equal to T^A.

The Hopf structure for $\mathcal{H}_R$ in terms of normal coordinates is derived by recalling the standard property of $\mathbf{C}$:

$$(\Delta \otimes \mathrm{id})\, e_*^{\tilde{\psi}^A \otimes Z_A} = e_*^{\Delta(\tilde{\psi}^A) \otimes Z_A} = e_*^{\tilde{\psi}^B \otimes \mathbf{1} \otimes Z_B} * e_*^{\mathbf{1} \otimes \tilde{\psi}^C \otimes Z_C}, \tag{19}$$

and applying the Baker–Campbell–Hausdorff (BCH) formula to the group product on the right-hand side of (19). Thus the coproduct $\Delta(\tilde{\psi}^A)$ is given by the coefficient of Z_A in the resulting Hausdorff series in the exponent. Explicitly

$$\Delta(\tilde{\psi}^A) = \tilde{\psi}^A \otimes \mathbf{1} + \mathbf{1} \otimes \tilde{\psi}^A + \tfrac{1}{2} f^A_{B_1 B_2} \tilde{\psi}^{B_1} \otimes \tilde{\psi}^{B_2} + \cdots. \tag{20}$$

Because the counit on T^A vanishes ($\epsilon(T^A) = 0$, $T^A \neq \mathbf{1}$), it clearly follows from (16) that the same is true for the $\tilde{\psi}^A$.

Recalling now that the exponential map is a bijection from $\partial\,\mathrm{Char}\mathcal{H}_R \rightarrow \mathrm{Char}\mathcal{H}_R$, and that the inverse of a character $\xi = \mathrm{e}^{\alpha^A Z_A}$ is given by $\xi^{-1} = \xi \circ S$, we can derive an expression for the action of the antipode S on the normal coordinates. Thus

$$\langle \mathrm{e}_*^{-\alpha^A Z_A}, \tilde{\psi}^A \rangle = \langle \xi^{-1}, \tilde{\psi}^A \rangle = \langle \xi \circ S, \tilde{\psi}^A) \rangle = \langle \xi, S(\tilde{\psi}^A) \rangle \tag{21}$$

and, since by the definition of normal coordinates

$$\langle \xi, \tilde{\psi}^A \rangle = \alpha^A, \tag{22}$$

it readily follows that

$$S(\tilde{\psi}^A) = -\tilde{\psi}^A. \tag{23}$$

3. Renormalization in terms of normal coordinates

The standard approach to renormalization in pQFT is to first regularize the theory by mapping the expressions corresponding to the Feynman diagrams onto analytic functions on the Riemann sphere $\mathbf{PC}^1$. In the Hopf algebra approach to renormalization this implies considering the homomorphisms from the algebra of decorated rooted trees, $\mathcal{H}_R$, to the unital $\mathbb{C}$-algebra $\mathcal{A} = \{f \in \mathrm{Holom}(\mathbb{C} - 0)\}$ with 0 a pole of finite order.

Let $\varphi, \varphi' \in \mathrm{Hom}_{\mathbb{C}\text{-alg.}}(\mathcal{H}_R, \mathcal{A})$ be two such linear maps $\varphi, \varphi' : \mathbb{C}(\mathcal{H}_R) \rightarrow \mathcal{A}$. Multiplication in this unital $\mathbb{C}$-algebra of homomorphisms is defined by the convolution product

$$(\varphi * \varphi')(T^A) = m_{\mathcal{A}}(\varphi \otimes \varphi')(\Delta T^A), \tag{24}$$

which correspondingly implies for the normal coordinates

$$(\varphi * \varphi')(\tilde{\psi}^A) = m_{\mathcal{A}}(\varphi \otimes \varphi')(\Delta \tilde{\psi}^A), \tag{25}$$

with the coproducts on the bi-algebras of rooted trees and normal coordinates defined, respectively, by admissible cuts of branches in rooted trees (for details cf e.g. [13]) and by equation (20) above. Note that by setting $\varphi(\tilde{\psi}^A)(z) := \psi^A \in \mathcal{A}$, $\varphi(T^A)(z) := \phi^A \in \mathcal{A}$, for $z \in \mathbb{C} - \{0\}$, we can make contact with the notation in [14].

In the Hopf algebra of renormalization the equivalent to the Forest formula is the twisted antipode axiom:

$$\phi_R^A = m_{\mathcal{A}} \circ (S_{\mathcal{R}} \otimes \mathrm{id})(\varphi \otimes \varphi)(\Delta T^A), \qquad \psi_R^A = m_{\mathcal{A}} \circ (S_{\mathcal{R}} \otimes \mathrm{id})(\varphi \otimes \varphi)(\Delta \tilde{\psi}^A). \tag{26}$$

Here ϕ_R^A and ψ_R^A stand for the renormalized ϕ^A and ψ^A, respectively, while $\mathcal{R}$ is the linear map $\mathcal{R} : \mathrm{Hom}_{\mathbb{C}\text{-alg.}}(\mathcal{H}_R, \mathcal{A}) \rightarrow \mathrm{Hom}_{\mathbb{C}\text{-alg.}}(\mathcal{H}_R, \mathcal{A})$, by $\varphi \mapsto \mathcal{R}(\varphi) := R \circ \varphi : \mathcal{H}_R \rightarrow R(\mathcal{A})$, and R is a Rota–Baxter projection operator, chosen to give the pole part of its argument (mass independent renormalization scheme), which satisfies the multiplicative constraints

$$R(ab) + R(a)R(b) = R\left(R(a)b + aR(b)\right), \qquad a, b \in \mathcal{A}. \tag{27}$$

This makes $\mathcal{A}$ a Rota–Baxter algebra of weight one. The multiplicative twisted antipode $S_{\mathcal{R}}$ is defined recursively by

$$\begin{aligned} S_{\mathcal{R}}(\phi^A) &= -R[\phi^A + m_{\mathcal{A}} \circ (S_{\mathcal{R}} \otimes \mathrm{id})(\varphi \otimes \varphi)\tilde{\Delta}(T^A)] \\ S_{\mathcal{R}}(\psi^A) &= -R[\psi^A + m_{\mathcal{A}} \circ (S_{\mathcal{R}} \otimes \mathrm{id}))\varphi \otimes \varphi)\tilde{\Delta}(\tilde{\psi}^A)]. \end{aligned} \tag{28}$$

In the above equations, the symbol $\tilde{\Delta}$ is used to denote the coproduct with the primitive part omitted.

Note that the target space of the counterterm map $S_{\mathcal{R}} : \mathcal{A} \to \mathcal{A}_-$ is

$$\mathcal{A}_- = \{\text{polynomials in } z^{-1} \text{ without constant term}\},$$

i.e. the principal part of the Laurent series for the ϕ^A or ψ^A, respectively.

Let us now apply the operator $(m_{\mathcal{A}} \otimes \mathrm{id}) \circ (S_{\mathcal{R}} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\varphi \otimes \varphi \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id})$ to equation (18) and make use of (19) and the commutativity of $\mathcal{A}$. We thus get

$$\begin{aligned}(\phi^B)_R \otimes Z_B &= \mathrm{e}_*^{S_{\mathcal{R}}(\psi^A)\otimes Z_A} * \mathrm{e}_*^{\psi^C \otimes Z_C} \\ &= \mathrm{e}_*^{[S_{\mathcal{R}}(\psi^A)+\psi^A+\frac{1}{2} f^A_{B_1 B_2} S_{\mathcal{R}}(\psi^{B_1})\psi^{B_2}+\cdots]\otimes Z_A} = \mathrm{e}_*^{(\psi^A)_R \otimes Z_A},\end{aligned} \tag{29}$$

where the term in the exponential in the equality before the last is the Hausdorff series for the BCH product.

4. Renormalization and poly-differential operators

Write (29) as

$$\mathrm{e}_*^V * \mathrm{e}_*^W = \mathrm{e}^{H(V,W)}, \tag{30}$$

where $V = S_{\mathcal{R}}(\psi^A) \otimes Z_A$, $W = \psi^C \otimes Z_C$. The Hausdorff series $H(V, W)$ has the properties

(i) $H = \sum_{n=1}^{\infty} H_n$, with H_n the homogeneous part of H of degree n in W given by [15]:

$$H_n = \frac{1}{n!}\left(H_1 \frac{\partial}{\partial V}\right)^n (V), \tag{31}$$

$$H_1 = W + \frac{1}{2}[V, W] + \sum_{k=1} \frac{B_{2k}}{(2k)!}\mathrm{ad}(V)^{2k}(W), \tag{32}$$

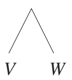
where B_{2k} are the Bernoulli numbers. Note however, that due to redundancies stemming from skew-symmetry and the Jacobi identities the resulting formula for the Hausdorff series obtained by this procedure, as well as for all other known presentations, has a non-minimal character;

(ii) $H(V, H(W, Z)) = H(H(V, W), Z)$ for $V, W, Z \in \mathcal{A} \otimes \mathfrak{L}$ (associativity);

(iii) $H(V, -V) = 0$, $H(V, 0) = H(0, V) = V$.

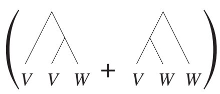
Consider next the free Lie algebra on the two generators V, W. By property (i) above, we can rewrite H as $H = \sum_{n=1}^{\infty} H^{(n)}$, where $H^{(n)}$ are now finite linear combinations of multi-commutators formed from n-letter words. Thus using the natural graphical encoding provided by Hall trees for Lie algebras [15], we have that each $H^{(n)}$ may be represented by a finite linear combination of Hall trees. Specifically

$$H^{(2)}(V, W) = \frac{1}{2}[V, W] = \frac{1}{2}$$



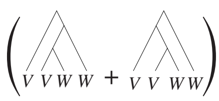
;

$$H^{(3)}(V, W) = \frac{1}{12}([V, [V, W]] + [[V, W], W]) = \frac{1}{12}$$



;

$$H^{(4)}(V, W) = \frac{1}{48}([V, [[V, W], W]] + [[V, [V, W]], W]) = \frac{1}{48}$$



;

and so on to higher orders.

If we now let $\mathfrak{L}^*$ denote the Lie algebra dual to $\mathfrak{L}$, it is well known that $\mathfrak{L}^*$ can be equipped with a linear Poisson structure, induced by the Lie bracket on $\mathfrak{L}$ [16], given by $f^A_{B_1 B_2} Z^*_A \partial_{B_1} \partial_{B_2}$, where the Z^*_A are coordinates on $\mathfrak{L}^*$ and the partials are taken relative to these coordinates.

Thus, starting with the BCH formula (30), we can introduce a $\star_1$-product for the algebra of exponential functions and rewrite (29) in the form

$$\begin{aligned} \mathrm{e}^{(\psi^A)_R Z^*_A} &= \mathrm{e}^{S_{\mathcal{R}}(\psi^B) Z^*_B} \star_1 \mathrm{e}^{\psi^C Z^*_C} \\ &:= \mathrm{e}^{S_{\mathcal{R}}(\psi^B) Z^*_B} \hat{D}\big(Z^*, f, (\overleftarrow{\partial}_{B_1}, \overleftarrow{\partial}_{B_2} \ldots), (\overrightarrow{\partial}_{C_1}, \overrightarrow{\partial}_{C_2} \ldots)\big)\, \mathrm{e}^{\psi^C Z^*_C}, \end{aligned} \tag{33}$$

where we have replaced the generators of the Lie algebra $\mathfrak{L}$, by the corresponding duals in $\mathfrak{L}^*$ and for notational simplicity the tensor sign between elements in $\mathcal{A}$ and elements in $\mathfrak{L}^*$ has been omitted, i.e. $\psi^C Z^*_C \in \mathcal{A} \otimes \mathfrak{L}^*$. In (33) $\hat{D}$ is the poly-differential operator:

$$\begin{aligned} &\hat{D}\big(Z^*, f, (\overleftarrow{\partial}_{B_1}, \overleftarrow{\partial}_{B_2} \ldots), (\overrightarrow{\partial}_{C_1}, \overrightarrow{\partial}_{C_2} \ldots)\big) \\ &\quad = \exp\bigg(\bigg\{\bigg[\frac{1}{2} \overleftarrow{\partial}_{B_1} f^{A_1}_{B_1 C_1} \overrightarrow{\partial}_{C_1} + \frac{1}{12}\big(\overleftarrow{\partial}_{B_1} \overleftarrow{\partial}_{B_2} f^{C_2}_{B_1 C_1} f^{A_1}_{B_2 C_2} \overrightarrow{\partial}_{C_1} \\ &\qquad + \overleftarrow{\partial}_{B_1} f^{A_2}_{B_1 C_1} f^{A_1}_{A_2 C_2} \overrightarrow{\partial}_{C_1} \overrightarrow{\partial}_{C_2}\big) + \cdots \bigg\}\bigg] Z^*_{A_1}\bigg). \end{aligned} \tag{34}$$

One can immediately conclude that the above $\star_1$-product is associative by computing the right side term in (33) and comparing with the BCH formula for the product of two group elements; the results are the same and, since the Hausdorff series is associative (by property (ii) above), so is the $\star_1$-product.

Moreover, the term on the right of the first equality in (33) is a sum of $\star_1$-products of polynomials on $\mathfrak{L}^*$ of the form $p_m \star_1 q_n$, $n, m = 0, 1, 2, \ldots$, which satisfy the properties:

- (A1) For any two polynomials p_m and q_n of orders m and n, respectively, the $\star_1$-product satisfies

$$p_n \star_1 q_m = p_n q_m + r_{n+m-1}, \tag{35}$$

  where $p_n q_m$ is the pointwise product of p_m and q_n, and r_{n+m-1} is a polynomial of degree $n + m - 1$.

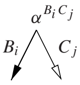
  This property follows from the fact that the poly-differential operator $\hat{D}$ in (34) reduces the total degree by one, except for the pointwise product resulting from the zeroth order in the expansion of the exponential in (34).

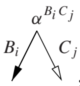
- (A2) from (A1) and the associativity of the $\star_1$-product it follows that for any two p_m and q_n the product $p_m \star_1 q_n$ is determined by knowing $(V)^n \star_1 W$, $n \in \mathbb{N}$, where without risk of confusion we now use the same symbols V, W to express $V = S_{\mathcal{R}}(\psi^A) Z^*_A$, and $W = \psi^C Z^*_C$, in terms of coordinates of $\mathfrak{L}^*$.

  The proof of this property is based on induction and a polarization identity. For further details of the proof we refer the reader to lemma 2.1.1 in [7].

Let us now consider the transition from Hall trees, encoding the Lie algebra in $H(V, W)$, to graphs for poly-differential operators coloured by a linear Poisson structure. This is discussed and illustrated extensively in [7], so here we shall only summarize the procedure which consists essentially in the following three steps:

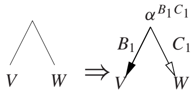
- Colouring the wedges in the Hall trees with the Poisson structure.

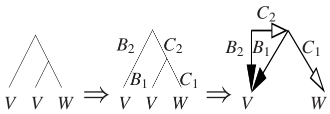
- Identifying the basic ordered wedge  with the bi-differential operator $\overleftarrow{\partial}_{B_i}(\alpha^{B_iC_j})\overrightarrow{\partial}_{C_j}$, i.e.

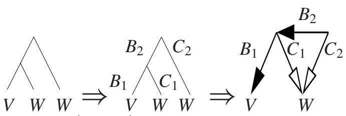
$$\overleftarrow{\partial}_{B_i}(\alpha^{B_iC_j})\overrightarrow{\partial}_{C_j} =$$  , (36)

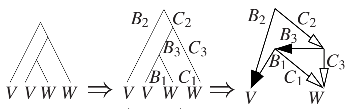
where $\alpha^{B_iC_j} \equiv \frac{1}{2} f^{A_i}_{B_iC_j} Z^*_{A_i}$ and where we assign the arrow $\longrightarrow\!\!\triangleright$ to the right action $\overrightarrow{\partial}_{C_j}$ and the black arrow $\longrightarrow\!\!\blacktriangleright$ to the left action $\overleftarrow{\partial}_{B_i}$.
- Merging all the V in the Hall trees into one point (to the left) and all the W into another point (to the right).

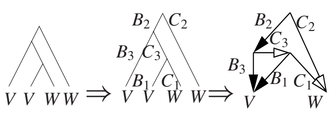
Thus for example to order $H^{(4)}$ in the Hausdorff series we get

$$[V, W] \equiv \mathrm{ad}_V(W) \Rightarrow$$  $$\Leftrightarrow \mathrm{e}^V\left(\overleftarrow{\partial}_{B_1} f^{A_1}_{B_1C_1} Z^*_{A_1} \overrightarrow{\partial}_{C_1}\right)\mathrm{e}^W$$
$$= \mathrm{e}^V\left(2\overleftarrow{\partial}_{B_1}\alpha^{B_1C_1}\overrightarrow{\partial}_{C_1}\right)\mathrm{e}^W; \tag{37}$$

$$[V, [V, W]] \equiv \mathrm{ad}^2_V(W) \Rightarrow$$ 
$$\Leftrightarrow \mathrm{e}^V\left(\overleftarrow{\partial}_{B_2}\overleftarrow{\partial}_{B_1} f^{C_2}_{B_1C_1} f^{D_2}_{B_2C_2} Z^*_{D_2}\overrightarrow{\partial}_{C_1}\right)\mathrm{e}^W$$
$$= \mathrm{e}^V\left((2)^2\overleftarrow{\partial}_{B_2}\overleftarrow{\partial}_{B_1}\left(\partial_{C_2}\alpha^{B_1C_1}\right)\alpha^{B_2C_2}\overrightarrow{\partial}_{C_1}\right)\mathrm{e}^W; \tag{38}$$

$$[[V, W], W] \equiv \mathrm{ad}_W(\mathrm{ad}_V(W)) \Rightarrow$$ 
$$\Leftrightarrow \mathrm{e}^V\left(\overleftarrow{\partial}_{B_1} f^{B_2}_{B_1C_1} f^{D_2}_{B_2C_2} Z^*_{D_2}\overrightarrow{\partial}_{C_1}\overrightarrow{\partial}_{C_2}\right)\mathrm{e}^W$$
$$= \mathrm{e}^V\left((2)^2\overleftarrow{\partial}_{B_1}\left(\partial_{B_2}\alpha^{B_1C_1}\right)\alpha^{B_2C_2}\overrightarrow{\partial}_{C_1}\overrightarrow{\partial}_{C_2}\right)\mathrm{e}^W; \tag{39}$$

$$[V, [[V, W], W]] \equiv \mathrm{ad}_V(\mathrm{ad}_W(\mathrm{ad}_V(W))) \Rightarrow$$ 
$$\Leftrightarrow \mathrm{e}^V\left(\overleftarrow{\partial}_{B_2}\overleftarrow{\partial}_{B_1} f^{B_3}_{B_1C_1} f^{C_2}_{B_3C_3} f^{D_2}_{B_2C_2} Z^*_{D_2}\overrightarrow{\partial}_{C_1}\overrightarrow{\partial}_{C_3}\right)\mathrm{e}^W$$
$$= \mathrm{e}^V\left((2)^3\overleftarrow{\partial}_{B_2}\overleftarrow{\partial}_{B_1}\left(\partial_{B_3}\alpha^{B_1C_1}\right)\left(\partial_{C_2}\alpha^{B_3C_3}\right)\alpha^{B_2C_2}\overrightarrow{\partial}_{C_1}\overrightarrow{\partial}_{C_3}\right)\mathrm{e}^W; \tag{40}$$

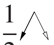
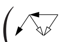
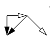
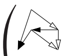
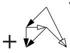
$$[[V, [V, W]], W] \equiv \mathrm{ad}_W\left(\mathrm{ad}^2_V(W)\right) \Rightarrow$$ 
$$\Leftrightarrow \mathrm{e}^V\left(\overleftarrow{\partial}_{B_3}\overleftarrow{\partial}_{B_1} f^{C_3}_{B_1C_1} f^{D_2}_{B_2C_2} f^{B_2}_{B_3C_3} Z^*_{D_2}\overrightarrow{\partial}_{C_1}\overrightarrow{\partial}_{C_2}\right)\mathrm{e}^W$$
$$= \mathrm{e}^V\left((2)^3\overleftarrow{\partial}_{B_3}\overleftarrow{\partial}_{B_1}\left(\partial_{C_3}\alpha^{B_1C_1}\right)\left(\partial_{B_2}\alpha^{B_3C_3}\right)\alpha^{B_2C_2}\overrightarrow{\partial}_{C_1}\overrightarrow{\partial}_{C_2}\right)\mathrm{e}^W. \tag{41}$$

In the above, summation over repeated indices is understood and, for clarity of the diagrams, the Poisson decorations of the vertices have been omitted.

It is evident from these illustrations, as well from equations (31)and (32) for H_n, that all the graphs for poly-differential operators resulting from the Hausdorff series are non-loop graphs such that a graph with n-vertices is formed by concatenation of a single wedge to a non-loop $(n-1)$-graph, allowing for the feet of the wedge to land either both on aerial vertices or one of them on a ground vertex and so that all the aerial vertices of the resulting n-graph have one leg from an ordered wedge landing on them, with the exception of the outermost aerial vertex. The category of these graphs is refered to in the literature as L-graphs.

Symbolically the poly-differential operator (34) can therefore be expressed as

$$\hat{D}\big(Z^*, f, (\overleftarrow{\partial}_{B_1}, \overleftarrow{\partial}_{B_2}\cdots), (\overrightarrow{\partial}_{C_1}, \overrightarrow{\partial}_{C_2}\cdots)\big)$$

$= \exp\Big[\frac{1}{2}$  $+ \frac{1}{12}\Big($  $+$  $\Big) + \frac{1}{48}\Big($  $+$  $\Big) + \cdots\Big].$ (42)

5. Relation to Kontsevich's quantization

In order to relate our preceding results with Kontsevich's $\star$-product for deformation quantization let us begin by reviewing briefly, both for self-containment and to fix notation, the essentials of that construction [3, 2].

Let $G_n, n \geqslant 0$ denote the class of admissible graphs, i.e. the class of oriented, labelled graphs $\Gamma \in G_n$ which diagrammatically are associated with all poly-differential operators that can be constructed from n wedges. Let V_Γ be the finite set whose elements are vertices of Γ, and E_Γ the finite set whose elements are the edges of Γ.

Definition 1. *An oriented graph* Γ *is a pair* (V_Γ, E_Γ) *such that* $E_\Gamma \subseteq V_\Gamma \times V_\Gamma$.

For $e = (v_1, v_2) \in E_\Gamma$ we say that the edge e starts at the vertex v_1 and ends at the vertex v_2.

A labelled graph Γ belongs to G_n if

- Γ has $n+2$ vertices and $2n$ edges.
- the set V_Γ is $\{1, 2, \ldots, n\} \cup L, R$, where L, R are just two symbols meaning left and right, and label the two ground vertices. These will correspond to the exponential functions $\exp(V)$ and $\exp(W)$ respectively, introduced in the previous section.
- there are two edges starting at every aerial vertex $k \in \{1, 2, \ldots, n\}$. These are ordered and labelled by e_k^1, e_k^2. In our notation of indexing with rooted trees, we have the correspondence $e_k^1 \leftrightarrow B_k, e_k^2 \leftrightarrow C_k$.
- for any $v \in V_\Gamma$, $(v, v) \notin E_\Gamma$, i.e. Γ has no loop (an edge starting at some vertex and ending at the same vertex). However, graphs with loops (also called wheels) formed by an edge starting at vertex i and ending at vertex j and another edge starting at vertex j and ending at vertex i are allowed ('valid' loops).
- for $n \geqslant 1$, the set G_n is finite and has $(n(n+1))^n$ elements, and one element for $n = 0$.

For an arbitrary Poisson structure $\alpha \in \Gamma\big(\mathcal{U}, \bigwedge^2 T_{\mathcal{U}}\big)$ in an open domain $\mathcal{U}$ of the space $\mathbb{R}^d$, which in coordinates $x^i, i = 1, \ldots, d$, on $\mathbb{R}^d$ is given by the bi-vector field $\alpha = \sum_{i,j=1}^d \alpha^{ij}(x)\partial_i \wedge \partial_j$, one can associate with each labelled graph $\Gamma \in G_n$ a poly-differential operator

$$B_{\Gamma,\alpha} : A \times A \to A, \qquad A = C^\infty(\mathcal{U}), \tag{43}$$

given by the general formula

$$B_{\Gamma,\alpha} = \sum_{I:E_\Gamma\to\{1,\ldots,d\}} \left(\prod_{e\in E_\Gamma, e=(*,L)} \overleftarrow{\partial}_{I(e)} \right) \left[\prod_{k=1}^{n} \left(\prod_{e\in E_\Gamma, e=(*,k)} \partial_{I(e)} \right) \alpha^{I(e_k^1)I(e_k^2)} \right] \times \left(\prod_{e\in E_\Gamma, e=(*,R)} \overrightarrow{\partial}_{I(e)} \right). \tag{44}$$

Here the map $I : E_\Gamma \to \{1, 2, \ldots, d\}$ (replacing the labels e_m^n by independent indices) corresponds to the colouring of edges used in the previous section, and is also useful in defining the colouring for the vertices.

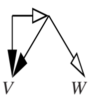
The next step in the Kontsevich construction consists in attaching a weight $w_\Gamma \in \mathbb{R}$ to each graph $\Gamma \in G_n$. The construction uses a special angular measure defined on the Poincaré plane. The actual computation of the weights for the most general graphs can become rather complicated and there are questions of uniqueness in the results from different calculation procedures [10].

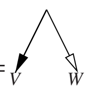
The associative Kontsevich $\star$-product is then defined as

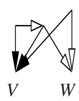
$$f \star g := f \left(\sum_{n=0}^{\infty} \varepsilon^n \sum_{\Gamma\in G_n} w_\Gamma B_{\Gamma,\alpha} \right) g, \tag{45}$$

where ε is the deformation parameter, and $f, g \in A$.

Our task is then to show that for a linear Poisson structure determined by the structure constants of the Lie algebra $\mathcal{L}$ introduced in section 2, equations (33) and (45) are the same and that the Kontsevich operator is of the form (42), i.e. the exponential of a sum of prime graphs (graphs that have no factors other than Γ_0 and themselves under multiplication). Thus composite graphs, resulting from multiplication of two prime graphs by merging their respective left and right ground vertices as in

$\Gamma_A =$ 

$\Gamma_B =$ 

$\Rightarrow \Gamma_{AB} =$ 

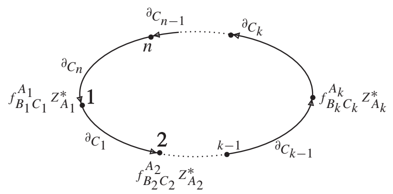
(46)

occur only through the expansion of the exponential.

To this end let us first set $\varepsilon = 1$ in (45). Also, since after regularization our normal coordinates are holomorphic functions on the Riemann sphere (cf beginning of section 3), and since the dual algebra $\mathcal{L}^*$ has infinite generators, we extend $A \to \mathcal{A} \otimes \mathcal{L}^* \simeq \mathcal{A} \otimes \mathbb{R}^\infty$ in (43), while differentiation is to be understood as acting on the space of coordinates Z_A^* on $\mathfrak{L}^*$. This poses no conceptual problem since the functions f, g occurring in (45) will now be exponentials where the exponents, for any tree with a finite number of vertices, will have only a finite number of terms contributing to the calculation. Next note that since we are dealing with a linear Poisson structure determined by the structure constants $f_{B_iC_j}^{A_l}$, there are no diagrams in the Kontsevich formula with two legs landing on any aerial vertex. Also note that valid loops (wheels) would involve cyclic products of the form

$$f_{B_1C_1}^{C_n} f_{B_2C_2}^{C_1} \cdots f_{B_nC_n}^{C_{n-1}}, \tag{47}$$

as may be seen from the generic loop



(48)

But by virtue of (5) one gets that

$$C_n > C_1 > C_2 > \cdots > C_{n-1} > C_n. \tag{49}$$

This is impossible (since it would imply that a tree is of a higher degree than itself), and in consequence there are no valid loop diagrams in the Kontsevich formula for the Poisson structure induced by $\mathfrak{L}$.

Thus, in the linear setting and our specific structure constants, the only admissible prime diagrams for the Kontsevich $\star$-product are non-loop L-graphs, as was the case for the BCH $\star_1$-product. So, in order to complete our argument that these two products are equivalent, irrespective of any particular expression for the Hausdorff series, we only need to show that the weights for their corresponding L-graphs are the same. The essentials for the proof are contained in theorem 5.0.2 in [7] (cf also [6, 10, 5]), which makes use of properties (A1) and (A2) discussed in the previous section.

Indeed, for two $\star$-products on $\mathfrak{L}^*$ to be equivalent all that is needed is

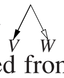
- to show that property (A1) is satisfied by both products, and
- to show that the two products are equal for products of the form $(V)^n \star W$.

The proof that $p_m \star q_n = p_m q_n$ + terms of degree $< m+n$ for the BCH product $\star_1$ that we considered in section 4 was already given there. To show it for the Kontsevich product (45) with the poly-differential (44) specialized to the linear setting is just as easy, as it only involves counting the legs landing on aerial vertices (which removes an equal number of powers of the Z^*_A from the vertex decorations) and the number of remaining legs landing on the two ground vertices (which reduces by an equal number the added degree $m+n$ of the p_m and q_n polynomials). For each type of the admissible diagrams the total degree is always lowered.

We have only left to show that for products of the form $(V)^n \star W$ the coefficients of the poly-differentials in the exponential of (42), taking into account our choice of Poisson structure, are the same as the corresponding ones in the summand of the Kontsevich operator:

$$\exp\left[\sum_{n=1}^{\infty}\sum_{\Gamma\in G_{n,L}} w_\Gamma B_{\Gamma,\alpha}\right], \qquad \text{where} \quad G_{n,L}\subset G_n \text{ is the subset of prime L-graphs.} \tag{50}$$

To this end, note that the diagrams associated with products of the form $(V)^n \star W$ are formed by succesive concatenations of the ad_V (

) graphs to the basic wedge, by attaching the V vertex of ad_V to the V vertex of 

and landing the other foot of ad_V on any of the free aerial vertices. As may be infered from (37), (38) and (42), the weights of these graphs for the BCH-quantization are given (cf equation (32)) by $2^n \frac{\tilde{B}_n}{n!}$, where $\tilde{B}_n = (-1)^n B_n$ and B_n are the Bernoulli numbers ($B_n = 0$ for $n > 1$ odd). Alternatively the Kontsevich weights for these graphs, calculated with the angle measures described in [7], are given by $w_\Gamma = \frac{\tilde{B}_n}{(n!)^2}$. But the diagrams, that we used in (42) to express $\hat{D}$ symbolically, are actually representatives of a class of diagrams of the same topological type which differ by the labelling of the vertices and edges. In fact, the class of topological type $[\Gamma_n]$ consists of $n!2^n$ distinguishable graphs (we can label the n vertices in $n!$ different ways, and there are two choices for the ordering of the two edges that emanate from each vertex). On the other hand the Γ that appear in the Kontsevich poly-differential are individual graphs and the summation is done over all individuals. But all the graphs belonging to a given topological class lead to the same poly-differential operator, both because the labelling of the vertices is irrelevant to the process of assigning poly-differential operators and because the change in sign resulting from the flipping of two edges from a given vertex (due to the antisymmetry of the α) is compensated by the change in sign of the weight factor of the graph [7]. We can therefore replace the sum over all Γ in the exponential of (50) by the sum of representatives of the corresponding topological class, with the *proviso* of multiplying each term in the chain by $n!2^n, n = 1, 2, \ldots, \infty$. Consequently the weights of the diagrams originating from products of the form $(V)^n \star W$, both in (42) and in (50) are the same. It then follows from the universality of the BCH and Kontsevich quantizations that the weights in both quantizations are the same for all the prime L-graphs. That is, for renormalized pQFT,

$$\exp\left[\sum_{n=1}^{\infty}\sum_{\Gamma\in G_{n,L}} w_\Gamma B_{\Gamma,\alpha}\right] = \hat{D}\big(Z^*, f, \big(\overleftarrow{\partial}_{B_1}, \overleftarrow{\partial}_{B_2}, \ldots\big), \big(\overrightarrow{\partial}_{C_1}, \overrightarrow{\partial}_{C_2}, \ldots\big)\big), \qquad G_{n,L} \subset G_n, \tag{51}$$

thus proving our contention.

Moreover, using (33) we have that

$$\exp((\psi^A)_R Z_A^*) = \exp(S_R(\psi^B)Z_B^*) \star \exp((\psi^C)Z_c^*), \tag{52}$$

where the Kontsevich $\star$-product is given by the operator $\exp\left[\sum_{n=1}^{\infty}\sum_{\Gamma\in G_{n,L}} \omega_\Gamma B_{\Gamma,\alpha}\right]$.

From (52) we can immediately infer that the renormalized normal coordinate $(\psi^A)_R$, corresponding to a given rooted tree labelled by the index A, is the coefficient of the coordinate Z_A^* which appears in the exponential after the $\star$-product on the right has been evaluated. Also, because of the commutativity of the Hopf algebras, and the associativity of their coproduct and that of the twisted antipode in the mass independent renormalization scheme [17], the renormalized $(\phi^A)_R$ (cf (26)) follow directly from (16). Thus

$$(\phi^A)_R = (\psi^A)_R + \sum_{m=2}^{\infty}\frac{1}{m!} n^A_{B_1\cdots B_m}(\psi^{B_1})_R\cdots(\psi^{B_m})_R. \tag{53}$$

6. The Birkhoff algebraic decomposition and the Kontsevich deformation

Our formulation of the Hopf algebra of renormalization in terms of normal coordinates together with equation (29) for the renormalized representative of a rooted tree provide an immediate relation with the Birkhoff algebraic decomposition and, in turn, a relation of the latter with the Kontsevich deformation.

In fact, as we have already seen from equation (28) in section 3, the twisted antipode projects the regularized normal coordinates $\psi^A \in \mathcal{H}_R$ onto $\mathcal{A}_-$. Therefore, defining

$$\phi_- := \mathrm{e}_*^{S_R(\psi^A)\otimes Z_A}, \qquad \phi := \mathrm{e}_*^{\psi^A\otimes Z_A}, \qquad \phi_+ := \mathrm{e}_*^{(\psi^A)_R\otimes Z_A}, \tag{54}$$

and substituting into (29) immediately yields the Birkhoff factorization:

$$\phi_+ = \phi_- * \phi. \tag{55}$$

Note, in particular, that

$$\langle \phi_-, (\mathrm{id} \otimes \psi^A)\rangle = S_{\mathcal{R}}(\psi^A), \qquad \langle \phi_+, (\mathrm{id} \otimes \psi^A)\rangle = (\psi^A)_R. \tag{56}$$

As a parenthetical remark, note also that we can relate the above results with those in [18] by writing $Z := \psi^B \otimes Z_B$ and $S_{\mathcal{R}}(\psi^B) \otimes Z_B \equiv -\mathcal{R}(\chi(Z)) \in \mathcal{A}_- \otimes \mathfrak{L}$, where

$$\chi(Z) = Z + \sum_{k=1}^{\infty} \chi_Z^{(k)}. \tag{57}$$

The $\chi_Z^{(k)}$ in the above series are derived by iteration on the equation:

$$\chi_Z^{(k)} = \sum_{i=1}^{k} c_i K^{(i)}\big(-\mathcal{R}\big(\chi_Z^{(k-i)}\big), Z\big), \qquad \chi_Z^{(0)} \equiv Z, \tag{58}$$

where $K^{(k)}(-\mathcal{R}(Z), Z)$ are the nested multicommutators of depth $k \in \mathbb{N}$ in the Hausdorff series, and the c_k their corresponding coefficients.

Thus calculating explicitly up to a depth 3 we have

$$\begin{aligned}
\chi_Z^{(1)} &= -\tfrac{1}{2}[\mathcal{R}(Z), Z],\\
\chi_Z^{(2)} &= \tfrac{1}{4}[\mathcal{R}([\mathcal{R}(Z), Z]), Z] + \tfrac{1}{12}\,([\mathcal{R}(Z), [\mathcal{R}(Z), Z]] + [Z, [\mathcal{R}(Z), Z]])\\
\chi_Z^{(3)} &= -\tfrac{1}{8}[\mathcal{R}([\mathcal{R}([\mathcal{R}(Z), Z]), Z]), Z] - \tfrac{1}{24}[\mathcal{R}(K^{(2)}(-\mathcal{R}(Z), Z)), Z]\\
&\quad + \tfrac{1}{24}K^{(2)}(\mathcal{R}([\mathcal{R}, Z], Z) + \tfrac{1}{48}K^{(3)}(-\mathcal{R}(Z), Z),
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
-\mathcal{R}(\chi(Z)) = -R\big\{&\psi^A - \tfrac{1}{2} f^A_{B_1C_1} R(\psi^{B_1})\psi^{C_1} + \tfrac{1}{4} f^A_{A_1C_2} f^{A_1}_{B_1C_1} R(R(\psi^{B_1})\psi^{C_1})\psi^{C_2}\\
&+ \tfrac{1}{12}\big(f^A_{B_1A_1} f^{A_1}_{B_2C_1} R(\psi^{B_1}) R(\psi^{B_2})\psi^{C_1} + f^A_{C_1A_1} f^{A_1}_{B_1C_2}\psi^{C_1} R(\psi^{B_1})\psi^{C_2}\big)\cdots\big\} \otimes Z_A.
\end{aligned} \tag{60}$$

The last expression is of course precisely the same as the one we would obtain for $S_{\mathcal{R}}(\psi^A)\otimes Z_A$ by making use of (28).

Also, making use of the BCH formula for the Hausdorff series together with (57) and (58), one has the following alternate expression for ϕ_+ (cf [18]):

$$\phi_+ = \mathrm{e}_*^{(1-\mathcal{R})(\chi(Z))}. \tag{61}$$

Let us now return to (55) with ϕ, ϕ_- and ϕ_+ given by (54) (or their equivalent expressions in terms of Z and $\chi(Z)$ given above). Comparing with the first equality in (29), they are clearly the same. Hence we can make use of (33) and (51) to conclude that

$$\begin{aligned}
\langle \phi_+, \mathrm{id} \otimes T^A\rangle \otimes Z_A &= \langle \phi_- * \phi, \mathrm{id} \otimes T^A\rangle \otimes Z_A\\
&\Leftrightarrow \mathrm{e}^{-\mathcal{R}(\chi(Z^*)}\exp\left[\sum_{n=1}^{\infty}\sum_{\Gamma\in G_{n,L}} w_\Gamma B_{\Gamma,\alpha}\right]\mathrm{e}^{Z^*},
\end{aligned} \tag{62}$$

where on the right-hand side we have used the notation $Z^* \equiv \psi^A Z^*_A$; $Z^*_A \in \mathfrak{L}^*$.

That is, renormalization of pQFT encoded in the Birkhoff algebraic decomposition can be viewed as a deformation, via the Kontsevich product $\phi_- \star \phi$, of the pointwise multiplication of the exponential functions ϕ_- and ϕ (expressed in terms of coordinates of the dual Lie algebra $\mathfrak{L}^*$, as was done in section 4), in the direction of the linear Poisson bracket.

7. Conclusions

In a previous paper [14], normal coordinates were introduced in the context of the Hopf algebra formalism of renormalization for the purpose of studying primitive elements of this algebra. It was shown there that the use of normal coordinates leads naturally to the concept of k-primitiveness, associated with the lower central series of the dual Lie algebra. For ladder trees with the same decoration on all vertices, it was also shown that normal coordinates provided remarkable simplifications in the renormalization process. Because of their specific relation to rooted trees (cf equation (18)) it is natural to expect that the ensuing simplified pole structure of the regularized normal coordinates, relative to that for rooted trees, will persist even for branched trees with multiple decorations on the vertices.

In the present paper we have shown that by further using the properties of the Hopf algebra of normal coordinates, a natural relation can be established between the twisted antipode axiom of renormalization (the Forest formula for pQFT) and the BCH quantization formula. We also showed the equivalence of this $\star$-product to the universal Kontsevich $\star$-product for deformation quantization, and that of the latter to the Birkhoff algebraic factorization. Last, but not the least, we showed that for pQFT the Kontsevich product is of the form of an exponential of a sum of weighted prime L-graphs.

There are other studies in the direction of establishing a connection between star products, Hopf algebras and quantum groups in field theory [19–23]. In particular in [21], the time-ordered product in field theory is related to the Weyl transform of a Drinfeld twisted product. For future work, it would be interesting to try to relate our work with these studies and see if from the combination of both approaches it is possible to obtain renormalized time ordered products and a more direct physical interpretation for the normal coordinates.

Acknowledgments

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